Guest Appearances of Generating Functions in Multitarget Tracking

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Outline

- Combinatorial problems
- Generating functions and functionals
- How they can arise in classical multitarget tracking filters
  - "Pointillist" filters
- Unifying continuous time formulation
- Future directions
- Conclusions

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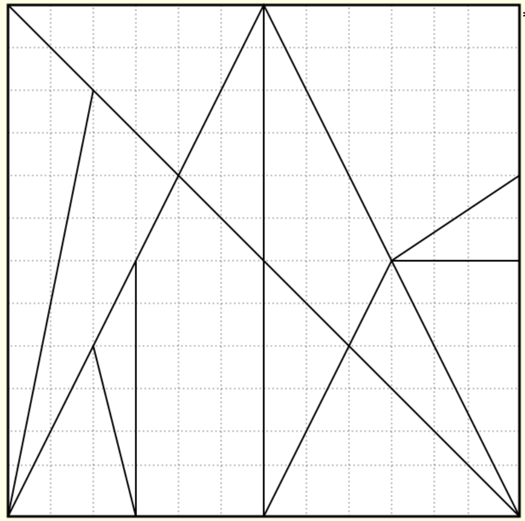
Combinatorial Problems are Ancient

- Stomachion
  14 piece
  dissection
  puzzle attributed
  to Archimedes
  (c. 287 BC – 212 BC)



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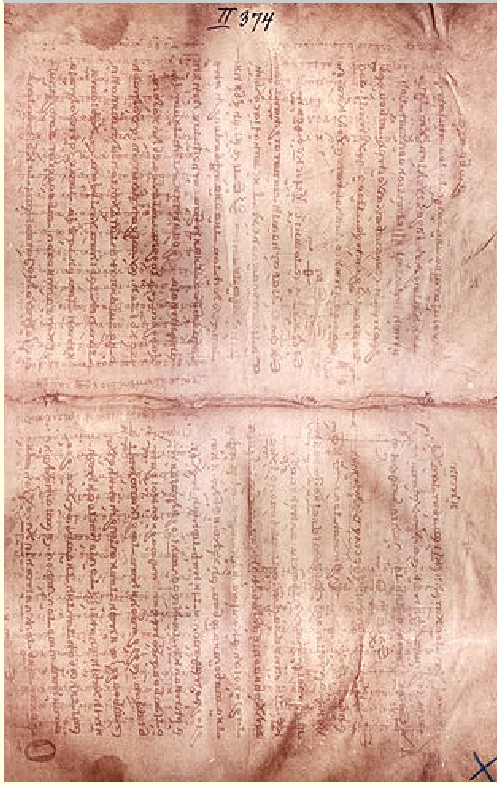
- Archimedes palimpsest
  - Unique, 10th century Byzantine parchment copy
  - *In 13th century it was scraped, washed, folded in half, turned 90 deg., and overwritten with liturgical text*
- How many ways can the 14 pieces be arranged into a square?
- Enumeration is the natural method
- Is there an analytical method?
- What is “analytic combinatics”?



The Walters Museum, Baltimore, MD USA

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Combinatorial Problems are Modern

- Prime Number Theorem. Asymptotic result about the "distribution" of prime numbers

$$\lim_{x \to \infty} \frac{\text{Number of primes } \leq x}{x / \ln x} = 1$$

- Conjectured independently by Gauss (1791, *at age 14*) and others (1797, Legendre, etc.)
- Proved by complex analytic methods (that involved the Riemann zeta function) in 1896, independently by Hadamard and de la Vallee-Poussin
- Began the field of "analytic number theory"
- "Elementary" arithmetic proof was discovered in 1949, independently by Selberg and Erdos

Probability Generating Function (PGF)

- Definition: $\mathbf{G}(s) = \sum_{a=0}^{\infty} \Pr\{a\}\, s^a$
  - s is called a "test variable"
  - Obviously, $\mathbf{G}(1) = 1$ (serves as a check)
  - Analytic in s on a complex domain that includes the disc $|s| \leq 1$
- Related functions of the probability sequence $\Pr\{a\}, a = 0,1,2,..$
  - z-transform: $\mathbf{G}(z^{-1})$
  - Moment generating function: $M(t) = \mathbf{G}(e^t)$
  - Characteristic function: $\Phi(\omega) = \mathbf{G}(e^{\omega\sqrt{-1}})$
- Key facts
  - PGF "*encodes*" probabilities. Differentiation "*decodes*" them:

$$\Pr\{a\} = \frac{1}{a!}\mathbf{G}^{(a)}(0) \equiv \frac{1}{a!}\frac{d^a}{ds^a}\mathbf{G}(s)\Big|_{s=0}$$

  - Mean number is $\mathbf{G}'(1) = \frac{d}{ds}\mathbf{G}(s)\Big|_{s=1}$
  - PGF of sum of *mutually independent* random integers is product of PGFs

Multivariate and Conditional PGFs

- Vector of random integers
  - Needs a different test variable for each vector component
- Bivariate case: Define $\mathbf{G}(x,y) = \sum_{a,b=0}^{\infty} \Pr\{a,b\}\, x^a y^b$

  → Analytic in each test variable on closed unit disc

  → Decoding: $\Pr\{a,b\} = \frac{1}{a!b!}\, \mathbf{G}^{(a,b)}(0,0)$

  → $\mathbf{G}(1,y)$ is the PGF of $\Pr\{b\} \equiv \sum_{a=0}^{\infty} \Pr\{a,b\}$

  → Decoding: $\Pr\{b\} = \frac{1}{b!} \mathbf{G}^{(0,b)}(1,0)$

- PGF of conditional distribution is normalized derivative

$$\mathbf{G}(x|b) = \frac{\mathbf{G}^{(0,b)}(x,0)}{\mathbf{G}^{(0,b)}(1,0)}$$

Bayes Theorem

  - Proof. $\mathbf{G}(x|b) = \sum_{a=0}^{\infty} \frac{\Pr\{a,b\}}{\Pr\{b\}}\, x^a = \frac{\textit{Taylor series in } x \textit{ about the point } x=0}{\mathbf{G}^{(0,b)}(1,0)}$

- Differentiate $\mathbf{G}(x|b)$ to decode the conditional probability distribution

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PGF of Histograms

- PGF of one sample drawn from four cells with probs. p_1, p_2, p_3, p_4

$$\mathbf{G}(x, y, z, w) = p_1 x + p_2 y + p_3 z + p_4 w$$

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- Histogram with exactly n i.i.d. samples is the product of their PGFs

$$\mathbf{G}(x, y, z, w | n) = (p_1 x + p_2 y + p_3 z + p_4 w)^n$$

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$$\mathbf{G}(x, y, z, w|n) = \boxed{(p_1 x + p_2 y + p_3 z + p_4 w)^n}$$

- Histogram with a random number of samples

$$\mathbf{G}(x, y, z, w) = \sum_{n=0}^{\infty} \Pr\{n\} \boxed{\sum_{\substack{a,b,c,d=0 \\ a+c+d=n}}^{\infty} \Pr\{a, b, c, d|n\}\, x^a y^b z^c w^d}$$

$$= \quad G^{\#}(p_1 x + p_2 y + p_3 z + p_4 w)$$

where $\mathbf{G}^{\#}(.)$ is PGF of the number *N* of histogram samples

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where $\mathbf{G}^{\#}(.)$ is PGF of the number N of histogram samples

- For M histogram cells with probabilities $p_{1:M}$ and test variables $x_{1:M}$

$$\boxed{\mathbf{G}(x_1, \dots, x_M) = \mathbf{G}^{\#}\left(\sum_{m=1}^{M} p_m\, x_m\right)}$$

Composition of PGFs

- PGF of N is on the "diagonal": $\mathbf{G}(s, \dots, s) = \mathbf{G}^{\#}\left(\sum_{m=1}^{M} p_m\, s\right) = \mathbf{G}^{\#}(s)$

Small Cell Limit

- As $M \to \infty$ (uniformly small cells on a bounded set S)

$$\mathbf{G}(x_1, \ldots, x_M) = \mathbf{G}^{\#}\left(\sum_{m=1}^{M} p_m\, x_m\right) \;\to\; \mathbf{G}^{\#}\left(\int_S\, p(s)x(s)\, ds\right)$$

  where the test variables $x_{1:M}$ become a test function $x(s), \quad s \in S$

  – PGF → PGFL (*Probability Generating FunctionaL*)

  – Bayesian PGF → Bayesian PGFL

  – Ordinary derivatives of the PGF → Functional derivatives of the PGFL

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  – Substitute a train of Dirac delta functions into PGFL
  – The ordinary derivatives with respect to the coefficients of the delta functions are the functional derivatives

Small Cell Limit

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- Can avoid functional derivatives of the PGFL
  - Substitute a train of Dirac delta functions into PGFL
  - The ordinary derivatives with respect to the coefficients of the delta functions are the functional derivatives
- Digression: You do not have to take the small cell limit
  - Discrete cells can be very useful for geo-spatial problems
    - Cells can be irregular in size and shape
    - Repeated measurements (counts) in any cell
  - Bayes PGF found using ordinary derivatives of the PGF
  - Easy intensity filter for discrete problem (Streit, IEEE T-AES, April 2014)

Applications to Tracking

Pointillist Filters

- *Definition* (Is this a serious definition?)

  A pointillist filter is a filter that is derived (exactly or by approximation) from a joint probability distribution that can be expressed in terms of a generating function.
- Examples:
  - Bayes-Markov
  - PDA
  - JPDA
  - IPDA (Integrated PDA, with track existence/nonexistence)
  - JIPDA (Joint IPDA)
  - PHD
  - CPHD
  - BAD (birth and death)
- General way to derive them all (*one ring to bind them* ☺)

Standard Bayes-Markov

- Prior state probability density function

  $\mu_0(x_0)$

- One target at x with probability of detection $P^D(x) = 1$
- Target motion (state transition) model

  $p(x \mid x_0)$

- Let $\mu(x) = \int \mu_0(x_0)\, p(x \mid x_0)\, dx_0$
- One measurement z and no clutter
- Measurement likelihood function:

  $p(z \mid x)$

- One test function for each variable: $x_0,\ x,\ z \iff h_0^A, h^A, h^B$
- Integrate over x_0 by setting $h_0^A = 1$

$$\Rightarrow \quad \mathbf{G}(1, h^A, h^B) = \int \int h^A(x)\, h^B(y)\, \mu(x)\, p(y \mid x)\, dy\, dx$$

- Note: $\mathbf{G}$ is a bilinear functional in h^A and h^B

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Standard Bayes-Markov (cont'd)

- $$\mathbf{G}(1, h^A, h^B) = \int\int h^A(x)\, h^B(y)\, \mu(x)\, p(y \mid x)\, dy\, dx$$
- Secular function: For n targets and m measurements, substitute

$$h^A(x) = \sum_{i=1}^{n} \alpha_i\, \delta(x - x_i)$$
$$h^B(x) = \sum_{j=1}^{m} \beta_j\, \delta(y - y_j)$$

- Bilinear algebraic form: $\mathbf{G}(\alpha, \beta) \equiv \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j\, \mu(x_i)\, p(y_j \mid x_i)$
- Avoid functional derivatives: use ordinary derivatives w.r.t. α and β
- Note: $\dfrac{d^n}{d\alpha_1 \cdots d\alpha_n} \mathbf{G}(\alpha, \beta) = 0$ for $n = 0$ and $n > 1$
  - As required, since there is exactly one target
- Similarly, $\dfrac{d^m}{d\beta_1 \cdots d\beta_m} \mathbf{G}(\alpha, \beta) \neq 0$ only for $m = 1$
  - As required, since there is exactly one measurement
- Exercise: show the same thing using functional derivatives

Standard Bayes-Markov (cont'd)

- *Only* one nontrivial secular function: $n = m = 1$
- To find PGFL of Bayes posterior using secular functions, define

$$\mathbf{G}(h^A, \beta_1) \equiv \mathbf{G}(h^A, \beta_1\, \delta(y - y_1)) = \beta_1 \int h^A(x)\, \mu(x)\, p(y_1 \mid x)\, dx$$

- PGFL of Bayes Theorem:

$$\mathbf{G}(h \mid y_1) \equiv \frac{\mathbf{G}^{(0,1)}(h^A, \beta_1)\Big|_{\beta_1=0}}{\mathbf{G}^{(0,1)}(1, \beta_1)\Big|_{\beta_1=0}} = \frac{\int h^A(x)\, \mu(x)\, p(y_1 \mid x)\, dx}{\int \mu(x)\, p(y_1 \mid x)\, dx}$$

- Check: $\mathbf{G}(1 \mid y_1) = 1$
- $\mathbf{G}(h^A \mid y_1)$ is a linear functional in h^A

  $\Leftrightarrow$ Probability density and intensity functions are identical

  Proof: $\mathbf{G}'(0 \mid y_1) = \mathbf{G}'(1 \mid y_1)$, where

$$\mathbf{G}(\alpha \mid y_1) = \mathbf{G}(\alpha\, \delta(x - x_1 \mid y_1) = \frac{\alpha\, \mu(x_1)\, p(y_1 \mid x_1)}{\int \mu(x)\, p(y_1 \mid x)\, dx}$$

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Bayes-Markov with Pd<1

$$\mathbf{G}(h^A, h^B) = \int h^A(x)\,\mu(x)\underbrace{\left[1 - P^D(x) + P^D(x)\int h^B(y)p(y \mid x)\,dy\right]}_{\substack{\textit{Measurement model} \\ \textit{Conditioned on } x}}\,dx$$

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Bayes-Markov with Pd<1

$$\mathbf{G}(h^A, h^B) = \int h^A(x)\,\mu(x)\underbrace{\left[1 - P^D(x) + P^D(x)\int h^B(y)p(y \mid x)\,dy\right]}_{\text{Measurement model Conditioned on x}} dx$$

$$= \underbrace{\int h^A(x)\,\mu(x)\left[1 - P^D(x)\right]dx}_{\text{Linear functional in } h^A} + \underbrace{\int h^A(x)\; h^B(y)\,P^D(x)\,\mu(x)\,p(y \mid x)\,dy\,dx}_{\text{Bilinear functional in } h^A \text{ and } h^B}$$

- Strictly speaking, $\mathbf{G}$ is not a bilinear form
  - Note. We will speak of such functions as if they are bilinear

PDA

- Assumptions
  - One target exists and generates at most one measurement
  - If the target is detected, the target-generated measurement is superposed with clutter (points of a Poisson point process)
  - Combinatorial problem – which point, if any, is the target?
- PGFL of the Bayes-Markov target:

$$\mathbf{G}(h^A, h^B) = \int h^A(x)\,\mu(x)\left[1 - P^D(x) + P^D(x)\int h^B(y) p(y \mid x)\,dy\right]dx$$

- PGF of Poisson random integer, mean λ_c : $\mathbf{G}^{\#}(s) = \exp(-\lambda_c + \lambda_c s)$
- PGFL of Poisson clutter with spatial PDF $\lambda(y)$ and mean λ_c

$$\mathbf{G}(h^B) = \mathbf{G}^{\#}\left(\int h^B(y)\,p(y)\,dy\right) = \exp\left(-\lambda_c + \lambda_c \int h^B(y)\,\lambda(y)\,dy\right)$$

- PGFL that leads to Bayes posterior distribution and PDA approx.

$$\mathbf{G}(h^A, h^B)\,\mathbf{G}(h^B)$$

- Secular function is product of a bilinear function and an exponential of a linear (affine) function

JPDA

- Assume that exactly *n* targets exist
- Each target requires its own test function and state space
- PGFL of the *i*-th Bayes-Markov target:

$$\mathbf{G}(h_i^A, h^B) = \int h_i^A(x)\,\mu(x)\left[1 - P^D(x) + P^D(x)\int h^B(y)p(y \mid x)\,dy\right]dx$$

- Targets and clutter are mutually independent
- Joint PGFL:

$$\mathbf{G}(h_1^A, \ldots, h_n^A, h^B) = \left[\prod_{i=1}^{n} \mathbf{G}(h_i^A, h^B)\right] \exp\left(-\lambda_c + \lambda_c \int h^B(y)\,\lambda(y)\,dy\right)$$

- Secular function is a product of *n* bilinear terms and an exponential of a linear term
- The cross-derivative of the secular function has exactly as many terms as there are feasible measurement-to-target assignments

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PHD and CPHD

- ONLY ONE TEST FUNCTION AND STATE SPACE FOR ALL TARGETS
  - Targets are assumed to be not identifiable
- Sharp contrast with JPDA
- The target process is independent of the clutter process
- PHD: Assume the PGF of the number of targets is Poisson distributed
- Joint PGFL:

$$
\begin{aligned}
&\mathbf{G}(h^A, h^B) \\
&= \exp\left[-\mu_{tar} + \mu_{tar} \int h^A(x)\,\mu(x)\left[1 - P^D(x) + P^D(x)\int h^B(y) p(y \mid x)\,dy\right]dx\right] \\
&\times \exp\left(-\lambda_c + \lambda_c \int h^B(y)\,\lambda(y)\,dy\right)
\end{aligned}
$$

- Secular function is exponential of sum of bilinear and linear functions
- CPHD: Assume the PGF of the number of targets is a polynomial of a specified maximum degree

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Other Closely Related Filters

- IPDA
  - Integrated PDA
- JIPDA
  - Joint IPDA

Continuous-Time Models for Target Number

- History: discrete-time birth and death (BAD) process
  - Bienaymé (1845), Galton-Watson (1874), Ulam-Hawkins (1944)
- Continuous-time analysis makes essential use of PGFs
  - Models considered here are classical (mid 20th century)
  - Birth and death with immigration (biological population growth)
- Begin with simple birth and death model, $N(t)$
  - State space is the set of natural numbers $\mathbb{N} = \{0,1,2,\ldots\}$
  - When the process is in state n, then
    - Time rates of target birth is $n\lambda$ and of target death is $n\mu$
  - Denote the test variable by s. For $t \geq 0$, the time-dependent PGF

    $$\mathbf{G}(s,t) = \sum_{n=0}^{\infty} \Pr\{N(t) = n \mid N(0) = 1\}\, s^n$$
  - Chapman-Kolmogorov (backward eqn.) → Diff. eqn. for $\mathbf{G}(s,t)$
  - Boundary condition at $t = 0$: One target initialization → $\mathbf{G}(s,0) = s$

Solution: $$\mathbf{G}(s,t) = \frac{\mu(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}{\lambda(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}$$

Check: $\mathbf{G}(1,t) = 1$

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Boundary Conditions

- Choice of boundary condition (BC) affects filter form and performance
- **BC**. $N(0) = n$ targets at time $t = 0$. Then $G(s, 0) = s^n$ and

$$\mathbf{G}(s,t) = \left(\frac{\mu(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}{\lambda(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}\right)^n$$

Exactly n targets are assumed mutually independent

  – Check: $\mathbf{G}(1, t) = 1$ for a (1,t)

  – $E[N(t)|N(0) = n] = \mathbf{G}'(0, t) = n\, e^{(\lambda-\mu)t}$

  – Pr{extinction}=$\left(\frac{\mu}{\lambda}\right)^n$. Example: 0.69 for $n = 2$ and 0.57 for $n = 3$

- **BC**. Poisson distributed number of targets at time $t = 0$. Then

$$\mathbf{G}(s,t) = \mathbf{G}^{\#}\left(\frac{\mu(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}{\lambda(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}\right)$$

where $\mathbf{G}^{\#}(s, 0) = e^{-\alpha + \alpha s}$ is PGF of Poisson random integer with mean α

  – $E[N(t)|N(0) = \alpha] = \mathbf{G}'(1, t) = \alpha\, e^{(\lambda-\mu)t}$

  – Mean matches the boundary condition with $N(0) = n$ if $\alpha = n$

    - Variance is larger, especially for small n
    - Probability of extinction is slightly larger (in the example)

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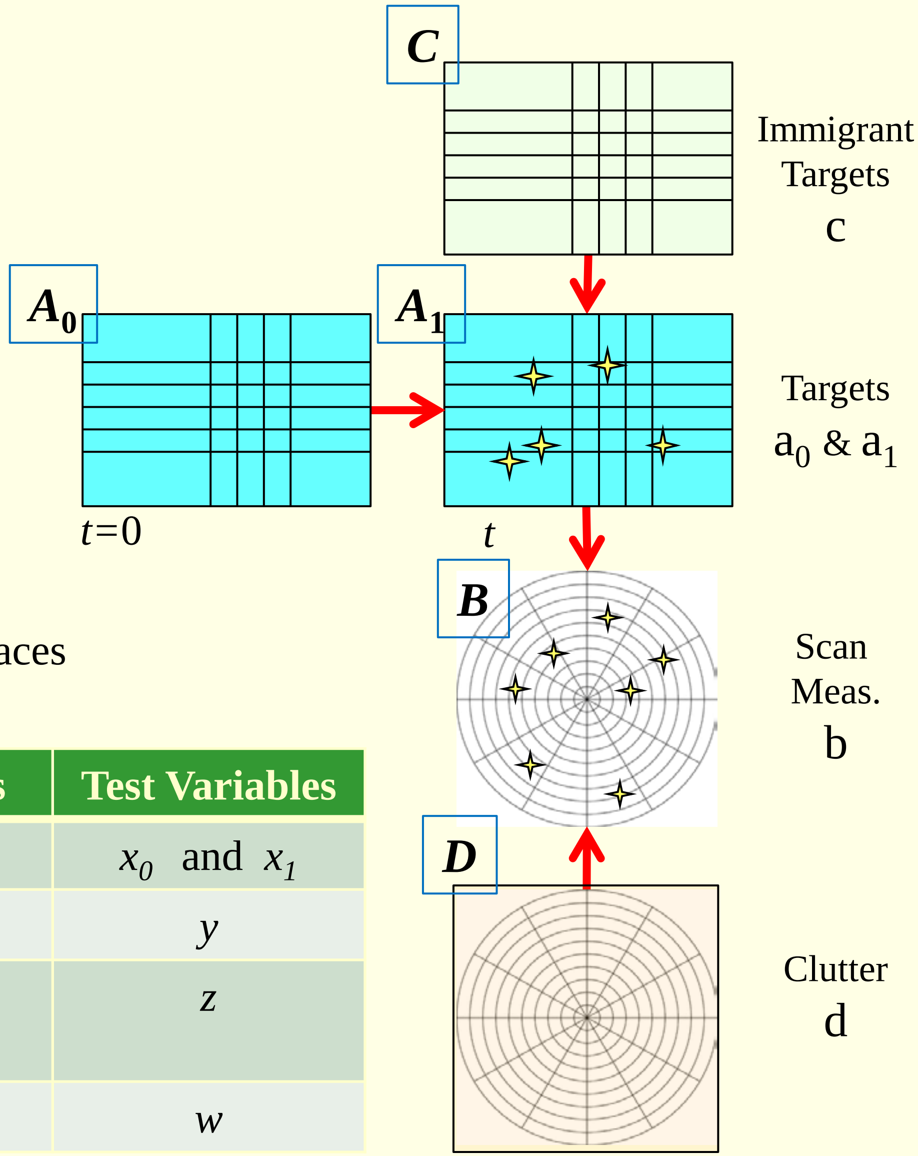
Number of Targets per Unit State Space

- Assume all spaces are bounded
- Partition into finite grids
- Take small cell limits
  - Sums → Riemann sums → Integrals
  - Test variables → Test functions
  - PGFs → PGFLs



- Assign test variable to every cell
  - Different names/labels for different spaces

Space	Grid Labels	Cell Counts	Test Variables
Target	A_0 and A_1	a_0 and a_1	x_0 and x_1
Measurement	B	b	y
Immigrant target	C	c	z
Meas. clutter	D	d	w

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General PGFL for Single Scan Problem

- PGFL: $\mathbf{G} = \mathbf{G}^{\#\mathbf{A}}\,\mathbf{G}^{\#\mathbf{C}}\,\mathbf{G}^{\#\mathbf{D}}$ where

$$\mathbf{G}^{\#\mathbf{A}}\left(\int_A \mu^A(x_0)h_0^A(x_0)\;\mathbf{G}^{\#\mathbf{A}|\mathbf{x}_0}\left(\int_A h_1^A(x)\,p_1^{A|A^-}(x \mid x_0)\,\mathbf{G}^{\#\mathbf{B}|\mathbf{x}}\left(\int_B h_1^B(y)p_1^{B|A}(y \mid x)\,dy\right)dx\right)dx^-\right)$$

$$\mathbf{G}^{\#\mathbf{C}}\left(\int_A \mu^C(z)h_1^C(z)\,\mathbf{G}^{\#\mathbf{A}|\mathbf{z}}\left(\int_A h_1^A(x)\,p_1^{A|C}(x \mid z)\,\mathbf{G}^{\#\mathbf{B}|\mathbf{x}}\left(\int_B h_1^B(y)\,p_1^{B|A}(y \mid x)\,dy\right)dx\right)dz\right)$$

$$\mathbf{G}^{\#\mathbf{D}}\left(\int_B \mu^D(w)h_1^D(w)\,\mathbf{G}^{\#\mathbf{B}|\mathbf{w}}\left(\int_B h_1^B(y)\,p_1^{B|D}(y \mid w)\,dy\right)dw\right)$$

  - Three prior PDFs: μ
  - Four conditional PDFs: p
  - Five test functions: h
  - Seven PGFs of integers: $\mathbf{G}^{\#}$

- Verify that $\mathbf{G} = 1$ when all test functions $= 1$

METRON SCIENTIFIC SOLUTIONS

General PGFL for Single Scan Problem

- PGFL: $\mathbf{G} = \mathbf{G}^{\#\mathbf{A}}\, \mathbf{G}^{\#\mathbf{C}}\, \mathbf{G}^{\#\mathbf{D}}$ where

$$\mathbf{G}^{\#\mathbf{A}}\left(\int_A \mu^A(x_0) h_0^A(x_0)\, \mathbf{G}^{\#\mathbf{A}|\mathbf{x}_0}\left(\int_A h_1^A(x)\, p_1^{A|A^-}(x \mid x_0)\, \mathbf{G}^{\#\mathbf{B}|\mathbf{x}}\left(\int_B h_1^B(y) p_1^{B|A}(y \mid x)\, dy \right) dx \right) dx^- \right)$$

$$\mathbf{G}^{\#\mathbf{C}}\left(\int_A \mu^C(z) h_1^C(z)\, \mathbf{G}^{\#\mathbf{A}|\mathbf{z}}\left(\int_A h_1^A(x)\, p_1^{A|C}(x \mid z)\, \mathbf{G}^{\#\mathbf{B}|\mathbf{x}}\left(\int_B h_1^B(y)\, p_1^{B|A}(y \mid x)\, dy \right) dx \right) dz \right)$$

$$\mathbf{G}^{\#\mathbf{D}}\left(\int_B \mu^D(w) h_1^D(w)\, \mathbf{G}^{\#\mathbf{B}|\mathbf{w}}\left(\int_B h_1^B(y)\, p_1^{B|D}(y \mid w)\, dy \right) dw \right)$$

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Birth and Death Models of Target Number

- Verify that **G** = 1 when all test functions = 1

PGFs of Number Can Be Simple Expressions

- Measurements generated by a target at x

$$\mathbf{G}^{\#\mathbf{B}|\mathbf{x}}\left(s \mid x\right) = 1 - \Pr^{D}(x) + \Pr^{D}(x)\, s \qquad \textit{at most one measurement / target}$$

- Immigrant target generators

$$\mathbf{G}^{\#\mathbf{C}}(s) = e^{-\nu + \nu s} \qquad \textit{Poisson, mean } \nu$$

- Immigrant targets given a target generator at z

$$\mathbf{G}^{\#\mathbf{A}|\mathbf{z}}(s \mid z) = s \qquad \textit{immigrant targets} \leftrightarrow \textit{generators}$$

- Clutter generators

$$\mathbf{G}^{\#\mathbf{D}}(s) = e^{-\lambda + \lambda s} \qquad \textit{Poisson, mean } \lambda$$

- Clutter points given a clutter generator at w

$$\mathbf{G}^{\#\mathbf{B}|\mathbf{w}}(s \mid w) = s \qquad \textit{clutter} \leftrightarrow \textit{generators}$$

- Predicted targets at time t given a target at x_0

$$\mathbf{G}^{\#\mathbf{A}|\mathbf{x}_0}(s \mid x_0) = 1 - \Pr^{Death}(x_0) + \Pr^{Death}(x_0)\, s$$

Boundary condition at $t = 0$: $\mathbf{G}^{\#\mathbf{A}}(s) = e^{-\bar{N} + \bar{N} s}$ *Poisson, recursive* $\bar{N}$

- Predicted targets at time t given a target at x_0

$$\mathbf{G}^{\#\mathbf{A}|\mathbf{x}_0}(s \mid x_0) = \frac{\mu(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}{\lambda(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)} \qquad \textit{Birth / Death} \begin{cases} \mu \equiv \mu^{death(x_0)} \\ \lambda \equiv \lambda^{birth(x_0)} \end{cases}$$

Boundary condition at $t = 0$: $\mathbf{G}^{\#\mathbf{A}}(s) = s^{n}$ *Recursive* n

METRON SCIENTIFIC SOLUTIONS

Bayesian Inference with PGFLs

- $\mathbf{G}\left(h_0^A, h_1^A, h_1^B, h_1^C, h_1^D\right)$ fully characterizes data and the model
- Given data, we can perform Bayesian inference on other variables
  - Marginalize by setting test functions = 1
- *Inference*
  - Targets at time t_1: $\mathbf{G}\left(h_1^A, h_1^B\right) \equiv \mathbf{G}\left(1,\ h_1^A, h_1^B,\ 1,\ \ 1\ \right)$
  - Clutter: $\mathbf{G}\left(h_1^D, h_1^B\right) \equiv \mathbf{G}\left(1,\ 1,\ h_1^B,\ 1,\ h_1^D\right)$
  - Immigrant targets: $\mathbf{G}\left(h_1^C, h_1^B\right) \equiv \mathbf{G}\left(1,\ 1,\ h_1^B, h_1^C,\ 1\ \right)$

May be new

- PGFL of Bayes posterior process is ratio of functional derivatives at $h_1^B = 0$
- Given m data points $\mathbf{y}=\{y(1),\ldots,y(M)\}$ in Y
  - $Example:$ PGFL for targets is $\mathbf{G}\left(h_1^A \,\middle|\, y(1),\ldots,y(M)\right) = \dfrac{\mathbf{G}_{\mathbf{y}}\left(h_1^A, 0\right)}{\mathbf{G}_{\mathbf{y}}\left(1, 0\right)}$

$$\mathbf{G}_{\mathbf{y}}\left(h_1^A, 0\right) \equiv \frac{\partial^M}{\partial\alpha(1)\cdots\partial\alpha(M)} \mathbf{G}\left(h_1^A, \sum_{m=1}^M \alpha(m)\,\delta_{y(m)}\right)\Bigg|_{\alpha(1)=\cdots=\alpha(M)=0}$$

Train of Dirac delta functions at the data

FUTURE DIRECTIONS
IN POINTILLIST TRACKING

Method

- Bayes
  - No need to generalize or extend the Bayesian method
  - Current Bayesian theory is satisfactory for tracking
- Pointillist multitarget tracking
  - Recall: Pointillist filters are derived from a Bayes posterior distribution that can be expressed in terms of a generating function
  - Need to explore the family of all such filters
- Connections with “multi-type” branching processes
  - “State” of the branching process is target number
  - “Type” is the traditional target state
- Age-dependent branching processes
  - Application to time between target maneuvers

Symbolic Differentiation

- Symbolic derivative expressions are too long
  - Even for Grad students
  - More time can be spent rendering the derivatives for visual inspection than in computing them
  - Transparent numerical evaluation of symbolic derivatives using Mathematica

- Symbolic derivatives are of limited utility in some problems

- Terms in the symbolic cross-derivative correspond one-to-one with the feasible assignments
  - Recall: A cross-derivative of a multivariate function f is a mixed derivative of f of order at most one with respect to any variable.
  - The combinatorics are encoded in the cross-derivative

Automatic Differentiation (AD)

- Possible role of automatic differentiation (AD)
  - AD is a mathematical method as well as a computer programming method
  - AD is a way to compute exact numerical values of symbolic derivatives
    - But ***without*** ever finding the symbolic derivative
  - Uses the chain rule.
    - *Example*: back-propagation in neural networks
- Theorem. All cross-derivatives up to order n can be evaluated with at most $O(n^2 2^n)$ computations
  - This is a general result.
  - Do special structures and symmetry in tracking help?
- AD is potentially useful in particle filter implementations
  - Particle weights can be found exactly using AD if the secular function of the generating function is known

METRON
SCIENTIFIC SOLUTIONS

Pair-Correlation Function

- Palm processes will be seen to be a statistically satisfying way to combine a detection process and a tracking process
- Potential for track extraction remains to be fully explored
  - Work with Bozdogan and Efe is on-going
- Exploit higher cross-derivatives with respect to target state
  - Within a given scan
  - Between successive scans

- Targets are assumed mutually independent.

  Question. So *why* is the pair-correlation function of the Bayes posterior process non-trivial, $\rho \neq 1$?
  - Bayes is optimal, yet it does not preserve mutual independence
  - Is it due to the imperfect (non-zero variance) sensing process?
  - Bayes posterior process corresponding to the PHD process is known to be repulsive, meaning that $\rho < 1$ (Bozdogan, Efe, Streit, Fusion 2013)
  - Is JPDA an attractive process, meaning that $\rho > 1$?

Training and Adaptation

- "Training" the distributions of the measurement and target processes
  - Pair-correlation functions will be critical elements
  - Very same kind functions appear when using EM (Expectation-Maximization) to estimate HMM parameters
    - from ensembles of short data sequences, or
    - one long sequence if HMM is stationary
  - They are also found in linear-Gaussian "batch" Kalman filter
    - Naturally, since it is the "small cell limit" of an HMM

- On-line Bayesian method for clutter and new target
  - Mentioned above for the single scan problem
    - Find the five-argument PGFL
    - For clutter, e.g., marginalize over targets not clutter

– Remain to be studied

Asymptotics and Approximations

- Direct approximations of PGFs and PGFLs
  - Hint of how this might be done in "approximate counting"
  - Mellin transforms
  - De-Poissonization

- Filters can be written as integrals in the complex domain
  - Asymptotics for large numbers of sensors
  - Saddle point methods (?)

- Approximations of this kind may open new applications for large numbers of sensors, data, etc.

New Applications

- Biological models will inspire new methods
  - Spatial-temporal modeling in epidemiology
  - Multi-type birth-death models are well matched to the phenomena
  - Measured geo-spatial data are integers

- Example: spread of cholera in Haiti
  - "*Leading Eigenvalues and the Spread of Cholera*"

    M. Gatto, L. Mari, and A. Rinaldo, SIAM News, Sept. 1, 2013

  - "*Generalized Reproduction Numbers and the Prediction of Patterns in Waterborne Disease*"

    M. Gatto, *et al*., Proc. Natl. Acad. Sci. USA, 48 (2012), 19703-19708.

Concluding Remarks

- Generating functions
  - Are useful in a broad family of tracking filters that have a strong combinatorial flavor that originates from assignment problems, e.g., measurement-to-target
  - Analytic combinatorics

- Not all combinatorial problems in tracking will yield to an analytic method

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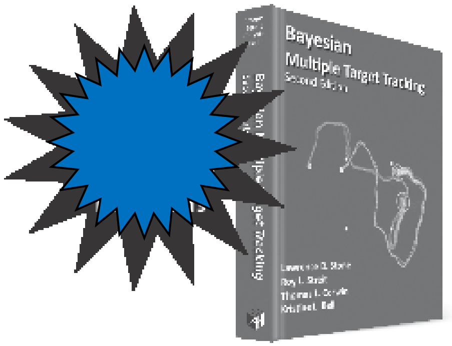

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